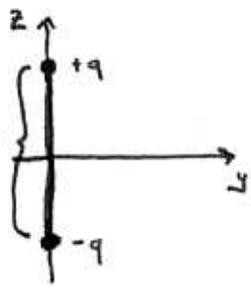


Physics 10B
Homework #8

#1. (Griffiths 11.3)

$$\vec{I}(t) = \frac{dq}{dt} \hat{z} = -q_0 \omega \sin \omega t \hat{z} \quad (\text{eq 11.15})$$

$$p_0 \equiv q_0 d$$



Thus,

$$P = I^2 R = q_0^2 \omega^2 \sin^2(\omega t) R \quad (\text{eq 11.15})$$

$$\Rightarrow \langle P \rangle_{\text{dissipated by wire}} = \frac{1}{2} q_0^2 \omega^2 R$$

Equating this to the average power radiated eq(11.22):

$$\Rightarrow \langle P \rangle_{\text{radiated}} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} = \frac{\mu_0 q_0^2 d^2 \omega^4}{12\pi c} = \frac{1}{2} q_0^2 \omega^2 R = \langle P \rangle_{\text{wire}}$$

$$\Rightarrow \boxed{R = \frac{\mu_0 d^2 \omega^2}{6\pi c}} \quad \text{since } \omega = \frac{2\pi c}{\lambda}$$

$$R = \frac{\mu_0 d^2}{6\pi c} \frac{4\pi^2 c^2}{\lambda^2} = \frac{2}{3} \pi \mu_0 c \left(\frac{d}{\lambda}\right)^2 = \frac{2}{3} \pi \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{d}{\lambda}\right)^2$$

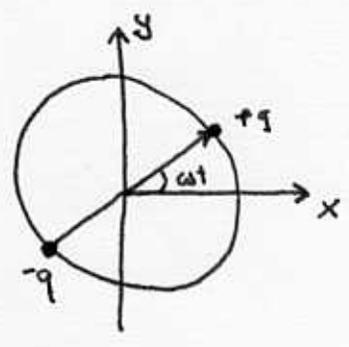
$$\Rightarrow \boxed{R = 789.6 \left(\frac{d}{\lambda}\right)^2 \Omega}$$

For wires in an ordinary radio w/ $d = .05 \text{ m}$ and say $\lambda = 10^3 \text{ m}$:

$$R = 789.6 \left(\frac{.05}{10^3}\right)^2 = 2 \times 10^{-6} \Omega \quad \text{which is negligible compared to wire resistances.}$$

#2. (Griffiths 11.4)

$$\vec{p} = p_0 (\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y})$$



For a single dipole on the z-axis we have the result given by (Eq. 11.18):

$$\textcircled{1} \quad \vec{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} \left(\frac{\sin\theta}{r} \right) \cos(\omega(t-r/c)) \hat{\theta} \quad (\text{Eq 11.18})$$

Since, $\hat{z} = \cos\theta \hat{r} - \sin\theta \hat{\theta}$ and, $\cos\theta = \frac{z}{r}$ in spherical coord.

thus, $\hat{\theta} = \left(\hat{z} - \frac{z}{r} \hat{r} \right) \left(-\frac{1}{\sin\theta} \right)$

Using this result in eq ① we get:

$$\vec{E} = \frac{\mu_0 p_0 \omega^2}{4\pi r} \cos(\omega(t-r/c)) \left(\hat{z} - \frac{z}{r} \hat{r} \right)$$

Now, by superposition the rotating dipole will give the following Electric field:

$$\vec{E} = E_x + E_y = \frac{\mu_0 p_0 \omega^2}{4\pi r} \left(\cos(\omega(t-r/c)) \left(\hat{x} - \frac{x}{r} \hat{r} \right) + \sin(\omega(t-r/c)) \left(\hat{y} - \frac{y}{r} \hat{r} \right) \right)$$

From eq (11.19) we have for a single dipole in the z-direction a magnetic field:

$$\vec{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} \left(\frac{\sin\theta}{r} \right) \cos(\omega(t-r/c)) \hat{\phi} \quad (\text{Eq 11.19})$$

$$= \frac{1}{c} (\hat{r} \times \vec{E}) \quad \text{by looking at eq ①}$$

Thus,

$$\vec{E}_{\text{total}} = \frac{\mu_0 P_0 \omega^2}{4\pi r} \left(\cos(\omega(t-r/c)) \left(\hat{x} - \frac{x}{r} \hat{r} \right) + \sin(\omega(t-r/c)) \left(\hat{y} - \frac{y}{r} \hat{r} \right) \right)$$

$$\vec{B}_{\text{total}} = \frac{1}{c} \left(\hat{r} \times \vec{E}_{\text{total}} \right)$$

Now, from the above results we can get the Poynting vector:

$$\textcircled{1} \quad \vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{1}{\mu_0 c} (\vec{E} \times (\hat{r} \times \vec{E})) = \frac{1}{\mu_0 c} (E^2 \hat{r} - (\vec{E} \cdot \hat{r}) \vec{E})$$

But,

$$\vec{E} \cdot \hat{r} = \frac{\mu_0 P_0 \omega^2}{4\pi r} \left(\cos(\omega t_r) \left(\hat{x} \cdot \hat{r} - \frac{x}{r} \right) + \sin(\omega t_r) \left(\hat{y} \cdot \hat{r} - \frac{y}{r} \right) \right)$$

$$\hat{x} \cdot \hat{r} = (\sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}) \cdot \hat{r} = \sin \theta \cos \phi$$

$$\hat{x} \cdot \hat{r} = x/r$$

$$\hat{y} \cdot \hat{r} = (\sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}) \cdot \hat{r} = \sin \theta \sin \phi$$

$$\hat{y} \cdot \hat{r} = y/r$$

Thus,

$$\vec{E} \cdot \hat{r} = 0 \quad \textcircled{2}$$

Placing eq $\textcircled{2}$ in eq $\textcircled{1}$ and taking the average we have:

$$\langle \vec{S} \rangle = \frac{\langle E^2 \rangle}{\mu_0 c} \hat{r}$$

Now, we need to find the Electric field squared:

$$\langle E^2 \rangle = \left(\frac{\mu_0 P_0 \omega^2}{4\pi r} \right)^2 \left\langle a^2 \cos^2(\omega t_r) + b^2 \sin^2(\omega t_r) + 2a \cdot b \sin(\omega t_r) \cdot \cos(\omega t_r) \right\rangle$$

$$\langle E^2 \rangle = \left(\frac{\mu_0 p_0 \omega^2}{4\pi r} \right)^2 \frac{1}{2} (a^2 + b^2) \quad (5)$$

$$\bar{a} \equiv \hat{x} - (x/r)\hat{r} \quad ; \quad \bar{b} \equiv \hat{y} - (y/r)\hat{r} \quad , \quad \text{thus:} \quad \hat{r} \cdot \hat{x} = x/r$$

$$a^2 = (\hat{x} - (x/r)\hat{r}) \cdot (\hat{x} - (x/r)\hat{r}) = 1 + \frac{x^2}{r^2} - 2\frac{x^2}{r^2} = 1 - \frac{x^2}{r^2}$$

$$b^2 = 1 - y^2/r^2$$

Placing the above results into eq (5):

$$\begin{aligned} \langle E^2 \rangle &= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r} \right)^2 \frac{1}{2} \left(2 - \frac{x^2 + y^2}{r^2} \right) \\ &= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r} \right)^2 \frac{1}{2} \left(2 - \frac{r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi}{r^2} \right) \\ &= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r} \right)^2 \left(1 - \frac{\sin^2 \theta}{2} \right) \end{aligned}$$

Thus,

$$\langle \bar{S} \rangle = \frac{1}{\mu_0 c} \left(\frac{\mu_0 p_0 \omega^2}{4\pi r} \right)^2 \left(1 - \frac{\sin^2 \theta}{2} \right) \hat{r}$$

$$\text{Intensity} = |\langle \bar{S} \rangle| = \frac{1}{\mu_0 c} \left(\frac{\mu_0 p_0 \omega^2}{4\pi r} \right)^2 \left(1 - \frac{\sin^2 \theta}{2} \right)$$

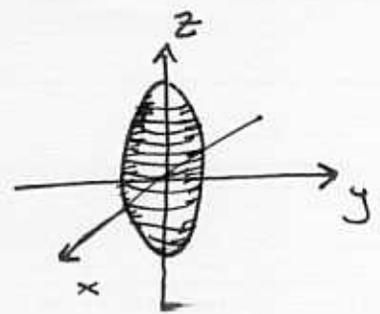
Power equals,

$$P = \int \langle \bar{S} \rangle \cdot d\bar{a} = \frac{\mu_0}{c} \left(\frac{p_0 \omega^2}{4\pi} \right)^2 \int \frac{1}{r^2} \left(1 - \frac{\sin^2 \theta}{2} \right) r^2 \sin \theta d\theta d\phi$$

$$= \frac{\mu_0 p_0^2 \omega^4}{16\pi^2 c} 2\pi \left(\int_0^\pi \sin \theta d\theta - \frac{1}{2} \int_0^\pi \sin^3 \theta d\theta \right) = \frac{\mu_0 p_0^2 \omega^4}{16\pi^2 c} 2\pi \left(2 - \frac{1}{2} \cdot \frac{4}{3} \right)$$

$$P_{\text{radiated}} = \frac{\mu_0 p_0^2 \omega^4}{6\pi c}$$

Intensity profile
 $I \sim 1 - \frac{1}{2} \sin^2 \theta$



For a single dipole the total power radiated equals

$$P_{\text{single dipole}} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \quad (\text{eq 11.22}), \text{ thus the power radiated}$$

for the superposition of two dipoles will be the two individual dipole powers plus any power from the cross term.

$$P_{\text{total}} = P_{x\text{-dipole}} + P_{y\text{-dipole}} + P_{\text{cross term}}$$

$$\begin{aligned} \vec{S}_{\text{total}} &= \frac{1}{\mu_0} (\vec{E}_T \times \vec{B}_T) = \frac{1}{\mu_0} ((E_x + E_y) \times (B_x + B_y)) \\ &= \frac{1}{\mu_0} (E_x \times B_x + E_y \times B_y + \underbrace{E_x \times B_y + E_y \times B_x}_{\text{cross terms}}) \end{aligned}$$

- In this particular case the components fields are 90° out of phase, so the cross terms go to zero, and the total power equals the sum of the two dipole powers.

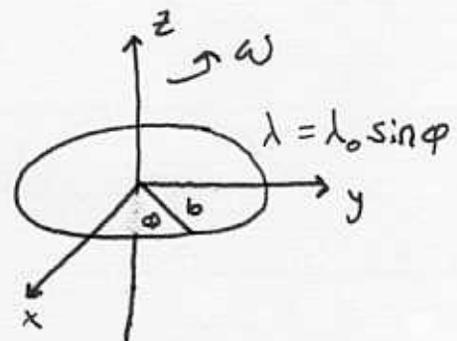
#3. (Griffiths 11.9)

At time $t=0$ the dipole moment of the ring is

$$\vec{p}_0 = \int \lambda r d\ell = \int \lambda_0 \sin\phi \hat{r} b d\phi$$

$$= \lambda_0 \int b \sin\phi (b \sin\phi \hat{y} + b \cos\phi \hat{x}) d\phi$$

$$= b^2 \lambda_0 \int_0^{2\pi} \sin^2\phi d\phi \hat{y} + b^2 \lambda_0 \int_0^{2\pi} \sin\phi \cos\phi d\phi \hat{x}$$



$$\bar{p}_0 = \pi b^2 \lambda_0 \hat{y}$$

As it rotates counterclockwise,

$$\bar{p}(t) = p_0 (\cos \omega t \hat{y} - \sin \omega t \hat{x})$$

So,

$$\ddot{\bar{p}} = -\omega^2 \bar{p}$$

Therefore, using (Eq. 11.60):

$$P = \frac{\mu_0 \ddot{p}^2}{6\pi c} = \frac{\mu_0}{6\pi c} \omega^4 p_0^2 = \frac{\mu_0}{6\pi c} \omega^4 \pi^2 b^4 \lambda_0^2$$

$$P = \frac{\pi \mu_0 \omega^4 b^4 \lambda_0^2}{6c}$$

#4. (Griffiths 11.13)

(a) Using the Larmor formula,

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \quad (\text{eq 11.70})$$

The time it takes to come to rest is $t = v_0/a$, so energy radiated equals,

$$U_{\text{rad}} = P t = \frac{\mu_0 q^2 a^2}{6\pi c} \frac{v_0}{a} = \frac{\mu_0 q^2 a v_0}{6\pi c}$$

The initial kinetic energy was,

$$U_{\text{kin}_0} = \frac{1}{2} M v_0^2 \Rightarrow \text{Thus, the fraction radiated} = \frac{U_{\text{rad}}}{U_{\text{kin}_0}} = \frac{\mu_0 q^2 a}{3\pi M v_0 c}$$

$$(b) \quad v_0 = 10^5 \text{ m/s} \quad d = 30 \text{ \AA}$$

From basic kinematics,

$$v_0^2 = 2ad \Rightarrow a = \frac{v_0^2}{2d}$$

Thus,

$$f = \frac{\mu_0 q^2}{3\pi M v_0 c} \frac{v_0^2}{2d} = \frac{\mu_0 q^2 v_0}{6\pi M c d} = \frac{(4\pi \times 10^{-7})(1.6 \times 10^{-19})^2 (10^5)}{6\pi (9.11 \times 10^{-31})(3 \times 10^8)(3 \times 10^{-9})}$$

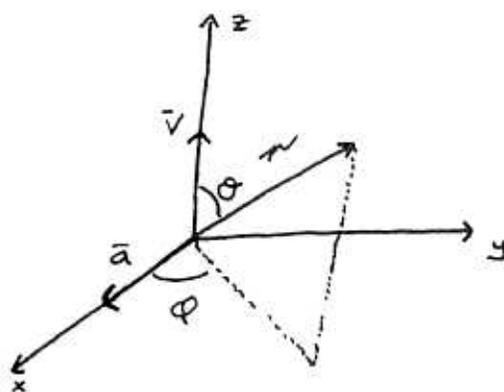
$$f = 2 \times 10^{-10}$$

So, the radiative losses in an ordinary wire are negligible.

#5. (Griffiths 11.16)

$$\vec{v} = v \hat{z}, \quad \vec{a} = a \hat{x}$$

$$\hat{r} = \sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}$$



The Lienard formula is:

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2 \epsilon_0} \frac{(\hat{r} \times (\vec{u} \times \vec{a}))^2}{(\hat{r} \cdot \vec{u})^5} \quad (Eq 11.72) \quad \beta \equiv \frac{v}{c}$$

$$\vec{u} = c \hat{r} - \vec{v} = c \hat{r} - v \hat{z}$$

$$\Rightarrow \hat{r} \cdot \vec{u} = c - v(\hat{r} \cdot \hat{z}) = c - v \cos\theta = c \left(1 - \frac{v}{c} \cos\theta\right) = c(1 - \beta \cos\theta)$$

$$\vec{a} \cdot \vec{u} = ac(\hat{x} \cdot \hat{r}) - av(\hat{x} \cdot \hat{z}) = ac \sin\theta \cos\phi$$

$$u^2 = \vec{u} \cdot \vec{u} = c^2 - 2cv(\hat{r} \cdot \hat{z}) + v^2 = c^2 + v^2 - 2cvc \cos\theta$$

Now, using the above results we can reduce the Lienard formula:

$$\hat{r} \times (\bar{u} \times \bar{a}) = (\hat{r} \cdot \bar{a}) \bar{u} - (\hat{r} \cdot \bar{u}) \bar{a}$$

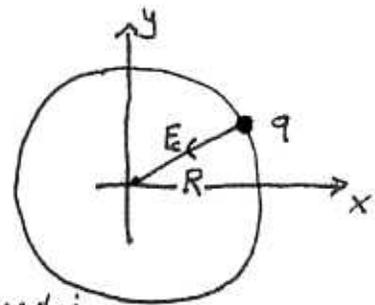
$$\begin{aligned} (\hat{r} \times (\bar{u} \times \bar{a}))^2 &= (\hat{r} \cdot \bar{a})^2 u^2 - 2(\bar{u} \cdot \bar{a})(\hat{r} \cdot \bar{a})(\hat{r} \cdot \bar{u}) + (\hat{r} \cdot \bar{u})^2 a^2 \\ &= (c^2 + v^2 - 2cv \cos \theta)(a \sin \theta \cos \phi)^2 - 2(ac \sin \theta \cos \phi)(a \sin \theta \cos \phi)c(1 - \beta \cos \theta) \\ &\quad + a^2 c^2 (1 - \beta \cos \theta)^2 \\ &= a^2 (c^2 (1 - \beta \cos \theta)^2 + (\sin^2 \theta \cos^2 \phi)(c^2 + v^2 - 2cv \cos \theta - 2c^2 + 2cv \cos \theta)) \\ &= a^2 c^2 ((1 - \beta \cos \theta)^2 - (1 - \beta^2) \sin^2 \theta \cos^2 \phi) \end{aligned}$$

Thus, placing the above results into the Lienard formula:

$$\frac{dP}{d\Omega} = \frac{q^2 a^2 c^2}{16\pi^2 \epsilon_0} \frac{((1 - \beta \cos \theta)^2 - (1 - \beta^2) \sin^2 \theta \cos^2 \phi)}{c^5 (1 - \beta \cos \theta)^5}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{((1 - \beta \cos \theta)^2 - (1 - \beta^2) \sin^2 \theta \cos^2 \phi)}{(1 - \beta \cos \theta)^5}$$

#6. (Griffiths 11.17)



(a) The radiation will apply a reaction force given by the Abraham-Lorentz formula:

$$\bar{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \dot{\bar{a}} \quad (\text{eq 11.80})$$

Thus, to counteract this force we need to apply a force,

$$\bar{F}_e = -\frac{\mu_0 q^2}{6\pi c} \dot{\bar{a}}$$

$$\vec{a}_{\text{cent}} = -\omega^2 \vec{r}$$

$$\Rightarrow \dot{\vec{a}} = -\omega^2 \vec{v}$$

So,

$$\boxed{\vec{F}_e = \frac{\mu_0 q^2}{6\pi c} \omega^2 \vec{v}}$$

The extra power delivered is,

$$P_e = \vec{F}_e \cdot \vec{v} = \boxed{\frac{\mu_0 q^2}{6\pi c} \omega^2 v^2 = P_e}$$

The power radiated from the Larmor formula is,

$$\text{(Eq. 11.70)} \quad P_{\text{rad}} = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{\mu_0 q^2}{6\pi c} \omega^4 r^2 = \frac{\mu_0 q^2}{6\pi c} \omega^2 v^2$$

Thus, we see that the two expressions are in fact equal $P_e = P_{\text{rad}}$

(b) For simple harmonic motion,

$$\vec{r}(t) = A \cos \omega t \hat{z}$$

$$\vec{v}(t) = \dot{\vec{r}} = -A\omega \sin \omega t \hat{z}$$

$$\vec{a}(t) = \dot{\vec{v}} = -A\omega^2 \cos \omega t \hat{z} = -\omega^2 \vec{r}$$

Thus,

$$\boxed{\vec{F}_e = \frac{\mu_0 q^2}{6\pi c} \omega^2 \vec{v}, \quad P_e = \frac{\mu_0 q^2}{6\pi c} \omega^2 v^2}$$

However, this time $a^2 = \omega^4 r^2 = \omega^4 A^2 \cos^2 \omega t$

Whereas $\omega^2 v^2 = \omega^4 A^2 \sin^2 \omega t$

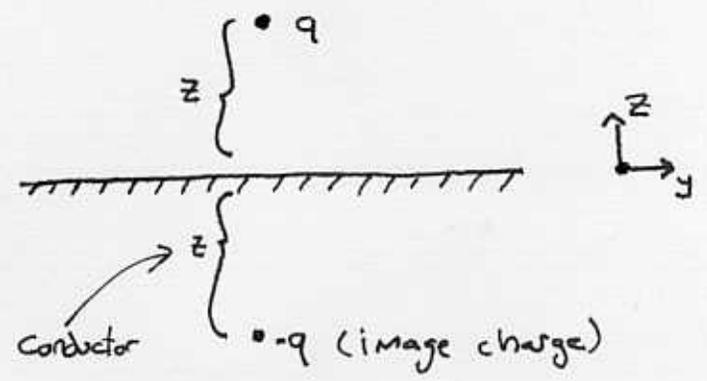
Thus,
$$P_{rad} = \frac{\mu_0 q^2}{6\pi c} \omega^4 A^2 \cos^2 \omega t \neq P_e = \frac{\mu_0 q^2}{6\pi c} \omega^4 A^2 \sin^2 \omega t$$

The power you deliver is not equal to the power radiated. However, since the time averages of $\sin^2 \omega t$ and $\cos^2 \omega t$ are $1/2$ over a full cycle the energy radiated is the same as the energy input.

#7. (Griffiths 11.25)

$$\vec{p}(t) = 2q\vec{z}(t)$$

$$\ddot{\vec{p}} = 2q\ddot{\vec{z}}$$



The charge above the conductor is attracted to the image charge through a Coulombic force,

$$F_{col} = M\ddot{z} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2z)^2}$$

$$\Rightarrow \ddot{z} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{4Mz^2} = -\frac{\mu_0 c^2 q^2}{16\pi M z^2}$$

$$\Rightarrow \ddot{\vec{p}} = -\frac{\mu_0 c^2 q^3}{8\pi M z^2}$$

Using (Eq 11.60), the power radiated is

(Eq 11.60)
$$P_{rad} = \frac{\mu_0 \ddot{\vec{p}}^2}{6\pi c} = \frac{\mu_0}{6\pi c} \left(-\frac{\mu_0 c^2 q^3}{8\pi M z^2} \right)^2$$

$$= \frac{\mu_0^3 c^3 q^6}{6(4\pi)^3 M^2 z^4} = \boxed{\left(\frac{\mu_0 c q^2}{4\pi}\right)^3 \frac{1}{6M^2 z^4} = P_{\text{rad}}}$$

#8. (Griffiths 11.31)

$$(a) \quad W_{\text{hyperbolic trajectory}} = \sqrt{b^2 + c^2 t^2} \quad (\text{Eq. 10.45})$$

$$P_{\text{rad}} = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c} \quad (\text{Eq. 11.75})$$

$$\text{So, } v = \dot{w} = \frac{c^2 t}{\sqrt{b^2 + c^2 t^2}}$$

$$a = \dot{v} = \frac{c^2}{\sqrt{b^2 + c^2 t^2}} - \frac{c^4 t^2}{(b^2 + c^2 t^2)^{3/2}} = \frac{c^2}{(b^2 + c^2 t^2)^{3/2}} (b^2 + c^2 t^2 - c^2 t^2)$$

$$= \frac{b^2 c^2}{(b^2 + c^2 t^2)^{3/2}}$$

$$\text{Now, } \gamma^2 = \frac{1}{1 - v^2/c^2} = \frac{1}{1 - c^2 t^2 / (b^2 + c^2 t^2)} = \frac{b^2 + c^2 t^2}{b^2 + c^2 t^2 - c^2 t^2}$$

$$= \frac{1}{b^2} (b^2 + c^2 t^2)$$

$$\text{So, } P_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \frac{b^4 c^4}{(b^2 + c^2 t^2)^3} \frac{(b^2 + c^2 t^2)^3}{b^6} = \boxed{\frac{q^2 c}{6\pi \epsilon_0 b^2} = P_{\text{rad}}}$$

Yes, a particle in hyperbolic motion radiates

$$(b) F_{\text{rad}} = \frac{\mu_0 q^2 \gamma^4}{6\pi c} \left(\dot{a} + \frac{3\gamma^2 a^2 v}{c^2} \right) \quad (\text{problem 11.30})$$

$$\dot{a} = -\frac{3}{2} \frac{b^2 c^2 (2c^2 t)}{(b^2 + c^2 t^2)^{5/2}} = -\frac{3b^2 c^4 t}{(b^2 + c^2 t^2)^{5/2}}$$

$$\text{Thus, } F_{\text{rad}} = \frac{\mu_0 q^2 \gamma^4}{6\pi c} \left(-\frac{3b^2 c^4 t}{(b^2 + c^2 t^2)^{5/2}} + \frac{3}{c^2} \frac{(b^2 + c^2 t^2)}{b^2} \frac{b^4 c^4}{(b^2 + c^2 t^2)^3} \frac{c^2 t}{\sqrt{b^2 + c^2 t^2}} \right)$$

$$\Rightarrow \boxed{F_{\text{rad}} = 0}$$

No, the radiation reaction is zero